Interaction of a soliton with a continuous wave packet

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We analyze the scattering of a soliton from a continuous wave packet of arbitrary shape in a fiber, theoretically as well as numerically. Solitons recover their original shapes and velocities after collisions and the effect of collisions is described by the change of velocities of solitons. The theoretical predictions based on a WKB-type approximation are in a good agreement with numerical results.

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One of the important problems in soliton communications is the soliton-continuum interaction. It was found that the collision of a wave packet with a soliton shows a change of the soliton position by the perturbing wave packet. This effect may be important in understanding phenomena such as the Gordon-Haus jitter [1], bit errors in lumped filters [2,3], and the pulse self-ordering in soliton fiber ring lasers [4,5]. Gordon [6] was the first to study the soliton-continuum system using the perturbed nonlinear Schrödinger equation. But the perturbation on the soliton was a second-order effect and did not show up in his study.

More general treatment of this problem, especially for the case of high-intensity wave packets, is shown in Ref. [7] based on the exact solution of the soliton-continuum wave of the nonlinear Schrödinger equation (NLSE). They analyze the scattering of a soliton from a continuous wave packet having a *constant* amplitude in a fiber, theoretically as well as numerically. Solitons were found to recover their original shapes and velocities after collisions, while shapes of continuous waves are nearly preserved during collisions. They describe the effect of collisions by the change of velocities of solitons and found that the theoretical predictions are in a good agreement with numerical results. This type of approach was further developed using another interesting solution of the NLSE, the collision of soliton-cnoidal wave, in Ref. [8], and was used to describe the soliton interaction with nonconstant continuous waves having moderately strong intensity.

In this respect, it is required to broaden the range of theoretical applicability of above approach to more general type of scattering phenomena, that is, the soliton scattering from *arbitrary-shaped* wave packets having *finite width*. In fact, a research along this line was already pursued by Haus *et. al.* [9]. They analyze the effect starting from the two-soliton solution and take the limit of (low intensity and) broad width of one soliton, and they infer the velocity change of the soliton to *arbitrary-shaped but low-intensity* wave packets. Note that the collision effect of two solitons has been described by the phase and time shift. So this notion was still preserved in scatterings of soliton from wave packets, which agrees with the results in Refs. [7,8].

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In this paper, we study the scattering of soliton with arbitrary-shaped wave packets, starting from the soliton-continuum solution. We introduce a WKB-type approximation for the theoretical interpretation of these scatterings. Our intention is to generalize the result of Ref. [9] such that it has no limitation of low-intensity wave packets. Thus, the main effect of collisions with a wave packet of general shape can be ascribed to the change of the velocity of solitons during collisions. This approximation explains numerical results very well except for wave packets having a rapidly varying amplitude or having very high intensity. Note that there exist no known exact solutions for collisions of solitons with arbitrary-shaped wave packets in the NLSE except the soliton-cnoidal wave (including the continuum) case.

We first start with numerical analyses. Figure 1 shows numerical results simulating typical collisions of solitons with wave packets having shape of (a) Gaussian type and (b) cosine type. These figures show that solitons restore their original shapes and velocities after collisions, even though they lose their identities during collisions. Despite their intrinsic instabilities [10], finite wave packets were also found to maintain their identities more or less.

The propagation of light waves in a fiber is described by the NLSE

$$i\frac{\partial}{\partial x}\psi+\frac{\partial^2}{\partial T^2}\psi+2|\psi|^2\psi=0, \tag{1}$$

where $T\equiv t-x/v_g$ is the retarded time. As an integrable equation, the NLSE has many important solutions. Most well known is the soliton solution,

$$\psi_{sol}(T,x)=A\ \operatorname{sech}(2Awx+AT)\times\exp(-iw^2x+iA^2x-iwT). \tag{2}$$

Another important one is the continuous wave,

$$\psi_{cw}(T,x)=p\ \exp(i[vT/2+2p^2x-v^2x/4]), \tag{3}$$

where v is the velocity of the continuous wave. Recently, there appears a nonlinearly superposed solution of (soliton + continuous wave) system [7] that was constructed using the Darboux-Bäcklund transformation (DBT) [11]. This solution was found to be convenient for application to real physical

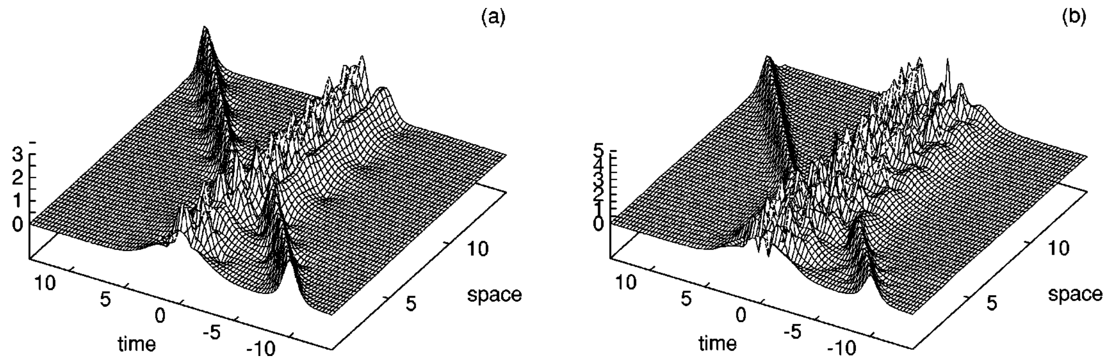

FIG. 1. Soliton scatterings from a finite (a) Gaussian, (b) cosine wave packet. They are obtained using the split-step fast-Fourier-transform algorithm.

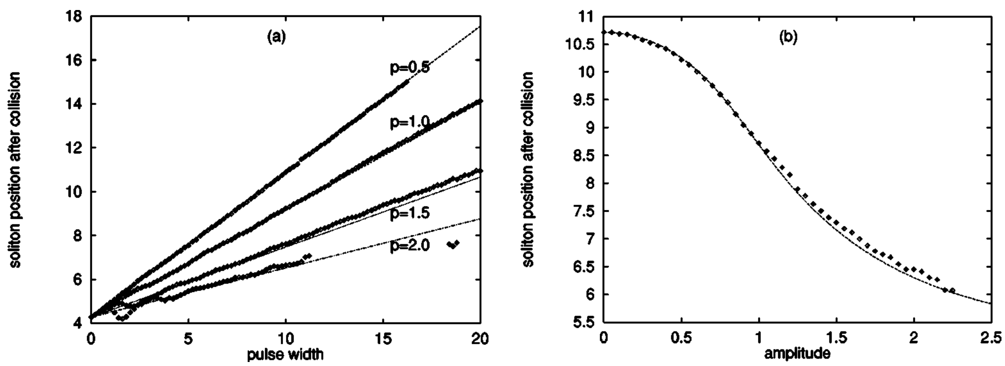

FIG. 2. Results from the finite Gaussian wave packet (a) Δx vs D for $p=0.5$, 1.0, 1.5, 2.0; (b) Δx vs p for $D=4.5$. Soliton parameters for both figures are $A=1.84$, $w=-0.7$.

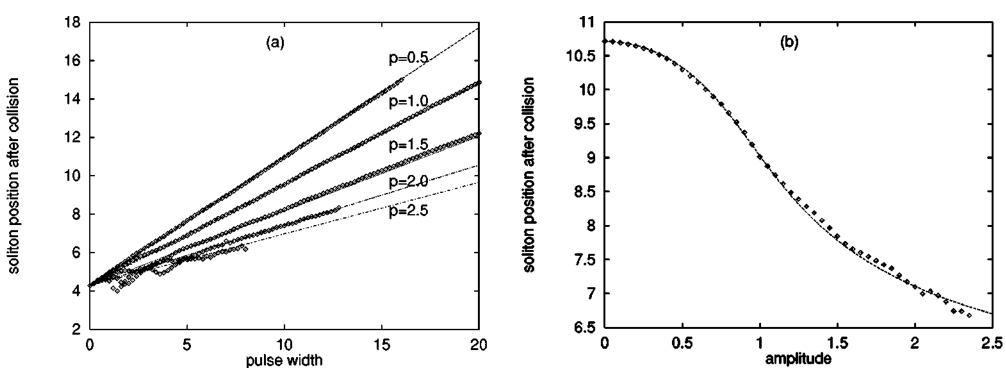

FIG. 3. Results from the finite cosine wave packet (a) Δx vs D for $p=0.5$, 1.0, 1.5, 2.0, 2.5; (b) Δx vs p for $D=4.5$. Soliton parameters for both figures are $A=1.84$, $w=-0.7$.

situations. Especially, its DBT parameter $-(w+iA)/2$ describes the amplitude A and the velocity of the soliton $v_g/(1-2v_g w)$ in Eq. (2) when it is located outside the continuous wave. The velocity of the soliton during its collision with the continuous wave changes to (for simplicity, we study $v=0, v_g=\infty$ case)

$$v_{sol+cw} = -\mathrm{Re}[\sqrt{(A-iw)^2-4p^2}]/\mathrm{Re}[\sqrt{(A-iw)^2-4p^2}(w+iA)]. \tag{4}$$

We now introduce a WKB-type approximation. It describes the velocity of a soliton in a continuous wave as

$$v_{WKB}(T) \sim v_{sol+cw}((p=p(T)), \tag{5}$$

when a continuous wave has slowly varying amplitude $p(T)$. Below, we will present some numerical results that show the validity of the WKB-type approximation. Figure 2 plots the moving distance Δx of a soliton in a time interval $-(D+\delta) \sim (D+\delta)$ passing through a Gaussian wave packet,

$$\begin{aligned} p(T) &= p \exp(-T^2/D^2) \quad &&\text{for } D<T<D,\\ &=0 &&\text{otherwise.} \end{aligned} \tag{6}$$

Below we will take $\delta=3$. The dotted curve is the result of numerical analyses. Theoretical values (solid line) are obtained using the WKB-type approximation such that

$$\Delta x = \int_{-D-\delta}^{D+\delta} v_{WKB}(T)\,dT. \tag{7}$$

Numerical results are in well accordance with theoretical values for wide range of D in the plot (a) of Δx vs D ($p=0.5,1,1.5,2$) and for wide range of p in the plot (b) of Δx vs p ($D=4.5$) without any fitting parameters [12]. Note that there are regions, for example, large D region for $p=2$ in Fig. 2(a), where the strong modulational instability of the continuous wave does not permit the identification of the soliton any more. Figure 3 plots the result from the cosine wave packet,

$$\begin{aligned} p(T) &= p\cos(\pi T/(2D)) \quad &&\text{for } -D<T<D,\\ &=0 &&\text{otherwise.} \end{aligned} \tag{8}$$

Figure 4 plots the result from the cnoidal wave packet,

$$p(T) = p\,\mathrm{dn}(pT,k). \tag{9}$$

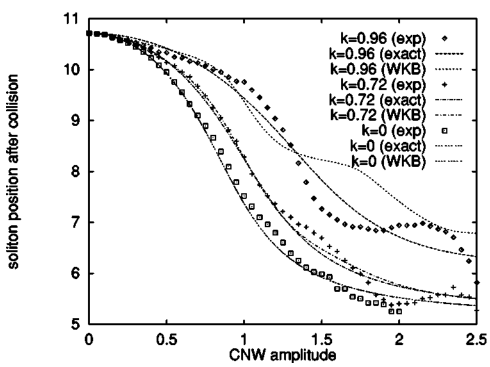

FIG. 4. Results from the finite cnoidal wave packet, Δx vs p for $k=0$, 0.72, 0.96 with $A=1.84$, $w=-0.7$.

Contrary to the previous cases, there exist exact theoretical results for the cnoidal wave packet. The numerical results (dotted curves) as well as the exact theoretical results are from Ref. [8] and we add the results from the WKB-type approximation. This figure shows that the exact theoretical calculation in Ref. [8] gives essentially the same results with the WKB-type approximation using Eq. (4) except for large k values ($k=0.96$). Note that the WKB-type approximation results in a large deviation from the exact calculation at $k=0.96$, because the amplitude of cnoidal wave fluctuates very fast and the WKB-type approximation becomes poor in this case (k: the modulus of the Jacobi function). On the other hand, the WKB-type approximation is in good accordance with the exact calculation as the modulus k becomes small, where the amplitude of cnoidal wave varies mildly and the WKB-type approximation works well. (At $k=0$, the cnoidal wave becomes a plain wave).[1]

Finally, Fig. 5 shows the average velocity

$$v_{av} = \frac{1}{2K}\int_{-K}^{K} v_{WKB}(T)\,dT \tag{10}$$

of a soliton inside the cnoidal wave calculated by the WKB-type approximation (solid line), which is compared with the exact result from Ref. [8] (dashed line) [13]. It reveals that the WKB-type approximation is in well accord with the exact results until $p\sim1.4$ for the case of $k=0.8$ or $k\sim0.6$ for the case of $p=1.5$, but the approximation shows small discrepancies from exact results for large amplitude p or for large modulus k.

[1]Here, the exact calculation means the result from analyses of (cnoidal wave+soliton) solutions of the NLSE, see Ref. [8]. Especially, the soliton velocity was obtained using a most important term and neglecting other unimportant terms [$\theta_i(\chi/2K)$ in Eq. (5) of Ref. [8]]. These neglected terms would have the effect of bending the curves in Fig. 4, especially at large k values. It results in discrepancies between plots of exact calculations and experimental values (typically, at large k).

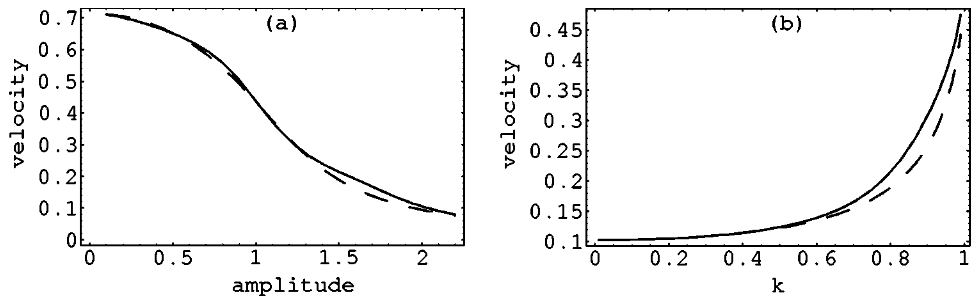

FIG. 5. (a) Soliton velocity vs amplitude p of cnoidal waves for $k=0.8$, (b) soliton velocity vs modulus k of cnoidal waves for $p=1.5$; solid lines from the WKB-type approximation, dashed line from the exact calculation.

Our expression for the soliton velocity in Eq. (4) becomes more simplified for the limit $p\rightarrow 0$ (here we restore v and v_g), which is

$$v+\frac{1}{v_g}-\frac{1}{v_{sol+cw}}=w_B\left(\frac{4p^2}{A^2+w_B^2}+\frac{16A^2p^4}{(A^2+w_B^2)^3}\right), \quad (11)$$

where $w_B=w+v/2$. Thus, the change in velocity of a soliton when it encounters a wave packet is

$$\Delta v_{sol}=v_{sol}-v_{sol}|_{p=0} =\frac{4w_Bp^2}{B^2(A^2+w_B^2)}\left(1+\frac{4A^2p^2}{(A^2+w_B^2)^2}+\frac{4w_Bp^2}{B(A^2+w_B^2)}\right), \quad (12)$$

where $B=1/v_g+v$. The first term of the right-hand side of the equation is very similar to the expression which was used to study the scattering of a soliton from a sinusoidal wave packet in the limit of $p\rightarrow 0$ [9]. Especially, it emphasizes on the fact that the change in velocity is proportional to the intensity of the wave packet ($=p^2$), i.e., a second-order effect. As our formula in Eq. (4) is valid even for large p, Eq. (12) can be used to estimate the error up to $O(p^4)$ in the results of Ref. [9].

All these considerations show the validity of the WKB-type approximation in describing the collision of the soliton-continuous wave packet. This description can be applied to the scattering of multisolitons from a continuous wave packet, too. In this case, each soliton experiences a change of the velocity during the collision with a continues wave packet, while the wave packet itself suffers no essential change. This configuration may account for interesting experimental observations such as forming of soliton clusters in a mode locked fiber ring laser [9]. Possible extension of our result to more general collisions such as the scattering of two continuous wave packets should involve the consideration of finite-gap solutions of the NLSE [14]. The difficulty in this program may be in treating the Riemann θ function as well as the so-called effectivization problem. The stability analysis of the collision process is another important problem to be done with exact solutions of the NLSE and is reserved for future study.

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